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Isochronicity of analytic systems via Urabe's criterion

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Abstract

A method for studying isochronous oscillations in some systems of ODE reducible to the equation $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$ is described. It is applied to obtain the necessary and sufficient conditions for isochronicity of a cubic two-dimensional autonomous system depending on six parameters. For all isochronous systems in this family the Urabe function is explicitly constructed.

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1. Introduction

Consider a planar autonomous analytic differential system of the form

$$\dot{x} = -y + \sum_{i+j=2}^{\infty} a_{ij} x^i y^j = -y + P(x, y), \qquad \dot{y} = x + \sum_{i+j=2}^{\infty} b_{ij} x^i y^j = x + Q(x, y). \tag{1}$$

Conversion to polar coordinates shows that near the origin either all non-stationary trajectories of (1) are ovals (in which case the origin is called a *centre*) or they are all spirals (in which case the origin is called a *focus*). In this paper we will study only systems with a centre at the origin. If all solutions near $x = 0$, $y = 0$ are periodic (that is, the origin is a centre), the problem then arises to determine whether the period of oscillations is constant for all solutions near the origin. A centre with such property is called *isochronous*. It follows from a result of Poincaré and Lyapunov that the centre of (1) is isochronous if and only if it is linearizable, that is, if there exists an analytic transformation $X = x + o(|x, y|)$, $Y = y + o(|x, y|)$ which brings (1) into the linear system $\dot{X} = -Y$, $\dot{Y} = X$.

Although the study of isochronous oscillations goes back at least to Huygens who investigated the motion of cycloidal pendulum, at present the problem is of renewed interest. Starting from the 1960s of last century many studies have been devoted to the investigation of the isochronicity and linearizability problems for various subfamilies of system (1).[4] We mention only very few contributions (for more references one can consult, e.g. [1, 5]). In 1964 Loud [11] classified isochronous centres of system (1) with P and Q being homogeneous polynomials of degree 2, and in 1969 Pleshkan [13] found all isochronous centres at the origin in family (1) where P and Q are homogeneous polynomials of degree 3. Sabatini [17], and Christopher and Devlin [7] have obtained efficient criteria for isochronicity of the Liénard system

$$\dot{x} = y, \qquad \dot{y} = -g(x) - f(x)y. \tag{2}$$

In [7] the authors also have classified all isochronous polynomial systems (2) of degree 34 or less.

The isochronicity problem and the properties of the period function of the Liénard type system,

$$\dot{x} = y, \qquad \dot{y} = -g(x) - f(x)y^2, \tag{3}$$

were studied by Sabatini [18]. Let us denote

$$F(x) = \int_0^x f(s)\,\mathrm{d}s, \qquad \phi(x) = \int_0^x \mathrm{e}^{F(s)}\,\mathrm{d}s. \tag{4}$$

It was shown in [18] that (3) can be transformed into the system

$$\dot{u} = y, \qquad \dot{y} = -g(\phi^{-1}(u))\,\mathrm{e}^{F(\phi^{-1}(u))} \tag{5}$$

by the substitution

$$u = \phi(x). \tag{6}$$

System (5) is a particular case of the system

$$\dot{x} = y, \qquad \dot{y} = -z(x). \tag{7}$$

Denoting $U(x) = \int_0^x z(s)\,\mathrm{d}s$ we obtain the first integral in the form 'kinetic energy + potential energy', that is, in the form

$$H(x, y) := \frac{y^2}{2} + U(x) = E, \tag{8}$$

where $H(x, y)$ is the Hamiltonian of (7). The following criterion of isochronicity of (7) is due to Urabe [20] (a simple proof of the criterion based on a formula from Landau and Lifshitz [10, p 25] is obtained in [15]).

Theorem 1 (Urabe's criterion)**.** *Assume $xz(x) > 0$ for $x \neq 0$ on an interval (a, b) containing the origin and $z(0) = 0, z'(0) = 1$. When $z(x)$ is continuous on (a, b), the necessary and sufficient condition that $z(x) \in C^1(a, b)$ and system (7) has an isochronous centre in the origin, is that, in a neighbourhood of $x = 0$ by the transformation*

$$\frac{1}{2}\xi^2 = \int_0^x z(s)\,\mathrm{d}s, \tag{9}$$

where $\xi/x > 0$ for $x \neq 0$, $z(x)$ is expressed as

$$z(x) = z[x(\xi)] = \frac{\xi}{1 + h(\xi)}, \tag{10}$$

where $h(\xi)$ is a continuous odd function.

[4] An interesting direction of research is also the study of the quantum spectrum of isochronous potentials, see e.g. [8] and the references therein

We call the function $h(\xi)$ defined in the theorem the *Urabe function*. For some properties and the meaning of this function the reader can consult [8].

It was first observed in [6] that the transformation (6) along with theorem 1 yield a method to compute the necessary conditions for isochronicity. In the present paper we describe the method of [6] in detail and show that it can be used in order to obtain conditions for isochronicity of some subfamilies of system (1), provided such subfamily depends on finite number of parameters and can be transformed to (3). Namely, we apply it to computing the conditions of isochronicity of the system

$$\begin{aligned} \dot{x} &= -y + axy + bx^2y \\ \dot{y} &= x + a_1x^2 + a_3y^2 + a_4x^3 + a_6xy^2, \end{aligned} \tag{11}$$

which is the so-called time-reversible system, that is, a system invariant under reflection with respect to a line passing through the origin and a change in the direction of time; due to this property all solutions of (11) in a neighbourhood of the origin are periodic. In theorem 3, which is the main result of the paper, we obtain the necessary and sufficient conditions for the isochronicity of the centre at the origin of system (11). To prove the sufficiency of the conditions we find explicitly the Urabe function $h(\xi)$. The obtained conditions are equivalent to those found by another method in [3, 14].

Usually in the studies of isochronicity problem for polynomial systems a variety of methods should be applied in order to check isochronicity (for instance, in [14] the authors used the Darboux method, commutativity and transformations preserving the period of oscillations to prove isochronicity in different subfamilies of system (11)). However, unexpectedly we found that in all cases of isochronicity of system (11) the Urabe function is of the form

$$h(\xi) = \alpha\xi/\sqrt{\beta^2 + \gamma\xi^2}. \tag{12}$$

We do not have an explanation of this fact. One possible hypothesis is that it is connected to the polynomial form of system (11). So, an interesting open question is whether there are systems (1) with polynomials on the right-hand side which can be reduced to (3) with the Urabe function not of the form (12)?

In the appendix, as a complement to the classification of the cubic system presented in [5], we give linearization transformations for few subfamilies of system (11).

2. An algorithm for computing conditions for isochronicity

Consider system (3):

$$\dot{x} = y, \qquad \dot{y} = -g(x) - f(x)y^2,$$

where $f(x)$ and $g(x)$ are analytic functions. The following theorem proven in [6] is the starting point of our work. It allows us to obtain an efficient algorithm for computing the necessary conditions for isochronicity of the system of the form (3) in the case when the coefficient of Taylor expansions of the functions $f(x)$ and $g(x)$ are polynomials of a finite number of variables which are parameters of (3), that is, when the functions are of the form

$$f(x) = \sum_{k=0}^{\infty} f_k(\alpha_1, \ldots, \alpha_n)x^k, \qquad g(x) = x + \sum_{k=2}^{\infty} g_k(\alpha_1, \ldots, \alpha_n)x^k. \tag{13}$$

Theorem 2 [6]. *Let f and g be functions analytic in a neighbourhood of the origin and $xg(x) > 0$ for $x \neq 0$. Then system (3) has an isochronous centre at the origin if and only if*

$$\frac{\xi}{1 + h(\xi)} = g(x)\,\mathrm{e}^{F(x)}, \tag{14}$$

where ξ *is defined by*

$$\frac{1}{2}\xi^2 = \int_0^x g(s)\,\mathrm{e}^{2F(s)}\,\mathrm{d}s, \tag{15}$$

and $h(\xi)$ *is an odd function such that*

$$\phi(x) = \int_0^x \mathrm{e}^{F(s)}\,\mathrm{d}s = \xi + \int_0^\xi h(t)\,\mathrm{d}t \tag{16}$$

and $\frac{\xi}{\phi(x)} > 0$ *for* $x \neq 0$.

Let us suppose that system (3) has a centre at the origin. We will deduce from theorem 2 an infinite set of conditions for isochronicity of the centre $(0, 0)$ in terms of the coefficients of the Taylor expansion of the functions f and g. Let h be the function defined by theorem 2 and $u = \phi(x)$, where ϕ is given by (4). Denote the left-hand side of (14) by $\tilde{g}(u)$,

$$\tilde{g}(u) = \frac{\xi}{1 + h(\xi)}. \tag{17}$$

If system (3) has a isochronous centre at the origin, then the function h must be odd,

$$h(\xi) = \sum_{i=1}^{\infty} c_{2i-1}\xi^{2i-1}. \tag{18}$$

From (17) and (18), one can find the kth derivative of $\tilde{g}(u)$, $\frac{\mathrm{d}^k\tilde{g}(u)}{\mathrm{d}u^k}$, by straightforward differentiation, $\frac{\mathrm{d}\tilde{g}(u)}{\mathrm{d}u} = \frac{\mathrm{d}\tilde{g}(\xi)}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}u}$, and, in general case

$$\frac{\mathrm{d}^k\tilde{g}(u)}{\mathrm{d}u^k} = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}^{k-1}\tilde{g}(u)}{\mathrm{d}u^{k-1}}\right)\frac{\mathrm{d}\xi}{\mathrm{d}u}. \tag{19}$$

On the other hand,

$$\tilde{g}(u) = \frac{\xi}{1 + h(\xi)} = g(x)\,\mathrm{e}^{F(x)}.$$

Thus, we can also find $\frac{\mathrm{d}^k\tilde{g}(u)}{\mathrm{d}u^k}$ by differentiating the function $g(x)\,\mathrm{e}^{F(x)}$ with respect to u. Namely,

$$\frac{\mathrm{d}\tilde{g}(u)}{\mathrm{d}u} = \mathrm{e}^{F(x(u))}\left(g(x)\frac{\mathrm{d}F}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}\right)\frac{\mathrm{d}x}{\mathrm{d}u}. \tag{20}$$

Since

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \mathrm{e}^{-F(x)}, \qquad \frac{\mathrm{d}F}{\mathrm{d}x} = f(x),$$

we obtain from (20)

$$\frac{\mathrm{d}\tilde{g}(u)}{\mathrm{d}u} = f(x)g(x) + \frac{\mathrm{d}g(x)}{\mathrm{d}x}.$$

By induction we see that

$$\frac{\mathrm{d}^k\tilde{g}(u)}{\mathrm{d}u^k} = \mathrm{e}^{(1-k)F(x)}S(x), \tag{21}$$

where $S(x)$ is a function of $f(x)$, $g(x)$ and their derivatives. Therefore, to compute the first m conditions for isochronicity of system (3) we can use the following algorithm.

- Fix $h(\xi) = \sum_{i=1}^{m} c_{2i-1}\xi^{2i-1} + O(\xi^{2m})$ and compute by (19) the kth derivative of $\tilde{g}(u)$ at 0,

$$v_k := \frac{\mathrm{d}^k \tilde{g}(0)}{\mathrm{d}u^k} \tag{22}$$

for $k = 1, \ldots, 2m+1$.
- By (21) compute

$$w_k := \frac{\mathrm{d}^k \bar{g}(0)}{\mathrm{d}u^k} \tag{23}$$

for $k = 1, \ldots, 2m+1$.
- Eliminate from the system

$$v_1 = w_1, \qquad v_2 = w_2, \ldots, v_{2m+1} = w_{2m+1}, \tag{24}$$

the variables $c_1, \ldots, c_{2m-1}$ to obtain a system

$$s_1 = s_2 = \cdots = s_m = 0, \tag{25}$$

which gives m necessary conditions for isochronicity of system (3).

Note that by increasing m we obtain, generally speaking, the infinite system of equations

$$s_1 = s_2 = s_3 = \cdots = 0. \tag{26}$$

If all these conditions are fulfilled then the corresponding system (3) has the isochronous centre at the origin. Any subset of (26) gives some necessary conditions for isochronicity of the centre. Let us emphasize that to obtain conditions (26) we do not use any explicit form of the Urabe function. We only assume its existence in the form of a series expansion with undetermined coefficients.

If (13) holds then s_k $(k = 1, 2, \ldots)$ are polynomials in $\alpha_1, \ldots, \alpha_n$. Thus, in such a case, by the Hilbert basis theorem (26) is equivalent to a system

$$s_1 = s_2 = s_3 = \cdots = s_M = 0, \tag{27}$$

$M \geqslant 1$. Unfortunately, neither Hilbert's theorem nor its proof gives any idea to obtain the number M appearing in (27). Thus we need some criteria to prove the equivalence of (26) and (27), that is to prove that (27) provides not only the necessary, but also the sufficient conditions for isochronicity. Two such criteria are given in the following section, and a simple (but we hope rather general) approach is described at the end of section 3.

3. Some criteria for isochronicity of system (3)

Before passing to the study of system (11) we mention some criteria of isochronicity of system (3).

Choosing special forms for the Urabe function one can obtain a number of criteria for isochronicity of system (3). For example, the following criterion corresponds to the case $h(\xi) \equiv 0$.

Criterion 1. *Let f and g be functions analytic in a neighbourhood of the origin and $xg(x) > 0$ for $x \neq 0$. If*

$$g'(x) + g(x)f(x) = 1, \tag{28}$$

then the origin is an isochronous centre of system (3).

Proof. If (28) holds, then

$$e^{F(x)} = e^{F(x)}(g'(x) + g(x)f(x)) = \frac{d}{dx}(g(x)\,e^{F(x)}),$$

which implies that $(g(x)\,e^{F(x)})^2 = \int_0^x 2g(s)\,e^{2F(s)}\,ds = \xi^2$. Thus, (14) holds and $h(\xi) \equiv 0$. By theorem 2, the centre at the origin is isochronous. □

Under the assumption that $f(x)$ and $g(x)$ are odd and of the class C^1 this criterion was obtained in [18]. The following statement corresponds to the case:

$$h(\xi) = \frac{k_1\xi}{\sqrt{k_2^2 + k_3\xi^2}}. \tag{29}$$

Criterion 2. *Assume that $f(x)$ and $g(x)$ are functions analytic in a neighbourhood of the origin and $xg(x) > 0$ for $x \neq 0$. If for some $k_1, k_2, k_3 \in \mathbb{R}$ and $k_2^2 + k_3^2 \neq 0$*

$$g'(x) + g(x)f(x) = (1 - k_1\Psi(x))^3 + k_1k_3\Psi^3(x), \tag{30}$$

where

$$G(x) = 2\int_0^x g(s)\,e^{2F(s)}\,ds, \qquad \Psi(x) = \frac{g(x)\,e^{F(x)}}{\sqrt{k_2^2 + k_3G(x)}},$$

then (3) has the isochronous centre at the origin.

Proof. When (30) holds, we have

$$\begin{aligned}
2g(x)\,e^{2F(x)} &= \frac{2g(x)\,e^{2F(x)}(g'(x) + g(x)f(x) - k_1k_3\Psi^3(x))(1 - k_1\Psi(x))}{(1 - k_1\Psi(x))^4}\\
&= \frac{((g(x)\,e^{F(x)})^2)'(1 - k_1\Psi(x))}{(1 - k_1\Psi(x))^4} - \frac{2g(x)\,e^{2F(x)}k_1k_3\Psi^3(x)(1 - k_1\Psi(x))}{(1 - k_1\Psi(x))^4}\\
&= \frac{((g(x)\,e^{F(x)})^2)'(1 - k_1\Psi(x))^2 + (1 - k_1\Psi(x))((g(x)\,e^{F(x)})^2)'k_1\Psi(x)}{(1 - k_1\Psi(x))^4}\\
&\quad - \frac{2g(x)\,e^{2F(x)}k_1k_3\Psi^3(x)(1 - k_1\Psi(x))}{(1 - k_1\Psi(x))^4}\\
&= \frac{((g(x)\,e^{F(x)})^2)'(1 - k_1\Psi(x))^2 - ((1 - k_1\Psi(x))^2)'(g(x)\,e^{F(x)})^2}{(1 - k_1\Psi(x))^4}\\
&= \left(\frac{(g(x)\,e^{F(x)})^2}{(1 - k_1\Psi(x))^2}\right)',
\end{aligned}$$

which implies that

$$\xi^2 = \int_0^x 2g(s)\,e^{2F(s)}\,ds = \frac{(g(x)\,e^{F(x)})^2}{(1 - k_1\Psi(x))^2}.$$

Then

$$\xi = \frac{g(x)\,e^{F(x)}}{1 - k_1\Psi(x)} = \frac{g(x)\,e^{F(x)}}{1 - \frac{k_1g(x)\,e^{F(x)}}{\sqrt{k_2^2 + k_3G(x)}}},$$

which means that (14) holds and $h(\xi)$ is of the form (29). By theorem 2, the centre at the origin is isochronous. □

Obviously, criterion 1 is a particular case of criterion 2 where $k_1 = 0$. Note also that criterion 2 can be derived from corollaries 2–7 of [6].

Example. To show how criterion 2 can be applied for studying isochronicity we consider the system

$$\dot{x} = y, \qquad \dot{y} = -x(1-x) - \frac{5y^2}{4(1-x)}. \tag{31}$$

For (31), the function $G(x)$ defined in criterion 2 is

$$G(x) = -2\left(4 + \frac{2(x-2)}{\sqrt{1-x}}\right).$$

Note that the coefficients k_1, k_2, k_3 in (29) are not uniquely defined, in fact, we have here a one-parametric family of the coefficients. Since the condition given by criterion 1 is not satisfied for (31) we can choose $k_1 = 1$. Setting now in equation (30) $x = -3, x = -8, x = -15$ (we have chosen these points in order to have simple expressions for $G(x)$ after evaluations) we obtain the system of three equations in two variables k_2 and k_3. Resolving it we find that $k_2^2 = 16$ and $k_3 = 1$. After some work for simplification of the expressions on the right-hand side of (30), we see that in a neighbourhood of the origin (30) is the identity when $k_1 = k_3 = 1, k_2 = 4$. Therefore, due to criterion 2 the centre at the origin of system (31) is isochronous and

$$h(\xi) = \frac{\xi}{\sqrt{16+\xi^2}} \tag{32}$$

is the Urabe function of the system.

Another way to see the isochronicity of the centre is the direct check that (32) is the Urabe function as follows. For system (31)

$$F(x) = \int_0^x f(s)\,\mathrm{d}s = -\frac{5}{4}\ln(1-x), \qquad g(x)\,\mathrm{e}^{F(x)} = x(1-x)^{-\frac{1}{4}}$$

and

$$\xi^2 = 2\int_0^x g(s)\,\mathrm{e}^{2F(s)}\,\mathrm{d}s = 2\int_0^x s(1-s)^{-\frac{3}{2}}\,\mathrm{d}s = 4\big[1-(1-x)^{\frac{1}{2}}\big]^2(1-x)^{-\frac{1}{2}}.$$

Thus,

$$\xi = 2\big[1-(1-x)^{\frac{1}{2}}\big](1-x)^{-\frac{1}{4}}, \qquad \sqrt{16+\xi^2} = 2(1-x)^{\frac{1}{4}} + 2(1-x)^{-\frac{1}{4}},$$

which imply that for $h(\xi)$ defined by (32)

$$\frac{\xi}{1+h(\xi)} = \frac{\xi\sqrt{16+\xi^2}}{\sqrt{16+\xi^2}+\xi} = \frac{4\big[1-(1-x)^{\frac{1}{2}}\big]\big[1+(1-x)^{-\frac{1}{2}}\big]}{2(1-x)^{\frac{1}{4}} + 2\big[2-(1-x)^{\frac{1}{2}}\big](1-x)^{-\frac{1}{4}}}$$
$$= x(1-x)^{-\frac{1}{4}} = g(x)\,\mathrm{e}^{F(x)}.$$

By theorem 2, the centre at the origin is isochronous.

4. Isochronicity of system (11)

We now demonstrate how the algorithm described in section 2 can be used in order to study the isochronicity problem for polynomial systems reducible to (3). Consider the cubic system (11):

$$\dot{x} = -y + axy + bx^2y$$
$$\dot{y} = x + a_1x^2 + a_3y^2 + a_4x^3 + a_6xy^2.$$

Differentiating the both sides of the first equation of (11) we see that system is equivalent to (3) with

$$f(x)=\frac{a+a_3+(2b+a_6)x}{1-ax-bx^2},\qquad g(x)=(x+a_1x^2+a_4x^3)(1-ax-bx^2).$$

Note, that $f(x)$ and $g(x)$ are functions with the property (13).

Theorem 3. *System (11) has an isochronous centre at the origin if and only if*

$$b=\left(-a^2+aa_1-10a_1^2+5aa_3-10a_1a_3-4a_3^2+9a_4+3a_6\right)/3$$

and one of the following conditions holds:

(1) $a_1+1/2a_3-1/2a=a_3a_4-1/6a_3a_6-3a_4a+1/6a_6a=a_3^2-3a_3a+2a^2-6a_4-4/3a_6$
$=a_3a_6a+6a_4a^2-a_6a^2-18a_4^2-a_4a_6+2/3a_6^2=a_4^2a^2+1/6a_4a_6a^2-3a_4^3+1/3a_4^2a_6$
$+5/36a_4a_6^2-1/54a_6^3=0,$
(2) $a_6=a_4=a_3-1/4a=a_1=0,$
(3) $a_6=4a_3-3a=a_1+a=a^2-3a_4=0,$
(4) $a_6=a_3-2a=a_1+2a=a^2-a_4=0,$
(5) $a_6=a_3-1/3a=a_1+2/3a=a^2-9/2a_4=0,$
(6) $a_4=a_1+1/2a_3-1/2a=a_3^2-3a_3a+2a^2-a_6=0,$
(7) $a_4=2a_3-a=2a_1+a=0,$
(8) $a_4=a_3-a=a_1=0,$
(9) $a_4=a_3=a_1=a^2-9a_6=0.$

Proof. To obtain the necessary conditions for isochronicity we use the algorithm of section 2 with $m=6$. To perform the first step of the algorithm we take

$$h(\xi)=\sum_{i=1}^{6}c_{2i-1}\xi^{2i-1}+O(\xi^{13})=d\xi+e\xi^3+c\xi^5+k\xi^7+t\xi^9+v\xi^{11}+O(\xi^{13}). \tag{33}$$

Computation of the derivatives by (19) yields

$$\begin{aligned}
&v_1=1,\quad v_2=-3d,\quad v_3=-15d^2,\quad v_4=-30e-105d^3,\\
&v_5=630ed+945d^4,\quad v_6=-840c-11\,340ed^2-10\,395d^5,\\
&v_7=30\,240cd+207\,900d^3e+11\,340\,e^2+135\,135d^6,\\
&v_8=-45\,360k-831\,600d^2c-623\,700d\,e^2-4054\,050d^4e-2027\,025d^7,\\
&v_9=21\,621\,600cd^3+34\,459\,425d^8+1663\,200ce+85\,135\,050d^5e+24\,324\,300d^2\,e^2\\
&\qquad+2494\,800dk,\\
&v_{10}=-567\,567\,000cd^4-654\,729\,075d^9-129\,729\,600cde-1929\,727\,800d^6e\\
&\qquad-851\,350\,500d^3\,e^2-16\,216\,200\,e^3-97\,297\,200d^2k-3991\,680t,\\
&v_{11}=86\,486\,400c^2+15\,437\,822\,400cd^5+13\,749\,310\,575d^{10}+6810\,804\,000cd^2e\\
&\qquad+47\,140\,493\,400d^7e+28\,945\,917\,000d^4\,e^2+1702\,701\,000d\,e^3\\
&\qquad+3405\,402\,000d^3k+194\,594\,400ek+311\,351\,040\,dt,\\
&v_{12}=-9081\,072\,000c^2d-439\,977\,938\,400cd^6-316\,234\,143\,225d^{11}\\
&\qquad-308\,756\,448\,000cd^3e-1237\,437\,951\,750d^8e-6810\,804\,000c\,e^2\\
&\qquad-989\,950\,361\,400d^5\,e^2-115\,783\,668\,000d^2\,e^3-115\,783\,668\,000d^4k\\
&\qquad-20\,432\,412\,000dek-16\,345\,929\,600d^2t-518\,918\,400v,
\end{aligned}$$

$$
\begin{aligned}
v_{13} = {} & 617\,512\,896\,000c^2d^2 + 13\,199\,338\,152\,000cd^7 + 7905\,853\,580\,625d^{12} \\
& + 13\,199\,338\,152\,000cd^4e + 34\,785\,755\,754\,750d^9e + 926\,269\,344\,000cd\,\mathrm{e}^2 \\
& + 34\,648\,262\,649\,000d^6\,\mathrm{e}^2 + 6599\,669\,076\,000d^3\,\mathrm{e}^3 + 57\,891\,834\,000\,\mathrm{e}^4 \\
& + 27\,243\,216\,000ck + 3959\,801\,445\,600d^5k + 1389\,404\,016\,000d^2ek \\
& + 741\,015\,475\,200d^3t + 32\,691\,859\,200et + 54\,486\,432\,000dv.
\end{aligned}
$$

Performing the second step of the algorithm we find

$$
\begin{aligned}
w_1 = {} & 1, \qquad w_2 = -a + 2a_1 + a_3, \\
w_3 = {} & -2b - 6a_1a + 6a_4 + (-2a + 2a_1)(a_3 + a) + 2a_6 + 2a^2 + 2a_3a \\
& - (-a + 2a_1 + a_3)(a_3 + a), \\
w_4 = {} & -16a_1b - 36a_4a - 2a_3a^2 + 4a_3b - 5a_6a - a^3 + 8ba + 12a_1aa_3 + a_3^2a + 4a_1a_6 \\
& + 14a_1a^2 - 2a_1a_3^2 - 7a_6a_3 - 12a_4a_3 + 2a_3^3, \\
w_5 = {} & a^4 - 30a^3a_1 + 5a^3a_3 - 70a^2a_1a_3 + 5a^2a_3^2 - 30aa_1a_3^2 - 5aa_3^3 + 10a_1a_3^3 - 6a_3^4 \\
& + 150a^2a_4 + 180aa_3a_4 + 30a_3^2a_4 + 9a^2a_6 - 10aa_1a_6 + 34aa_3a_6 - 30a_1a_3a_6 \\
& + 29a_3^2a_6 - 8a_6^2 - 22a^2b + 120aa_1b - 40aa_3b + 80a_1a_3b - 10a_3^2b - 120a_4b \\
& - 8a_6b + 16b^2, \\
w_6 = {} & 138a_1a_6a_3a + 62a_1a^4 - 1080a_1ba_3a + 63aa_6^2 - 25a_3^2a^3 - 46a_1a_6^2 + 208a_1a_6a_3^2 \\
& - 400a_1ba_3^2 + 90a_1a_3^3a + 92a_3a_6b + 350a_1a_3^2a^2 + 200a_3^2ab - 220a_6a_3^2a \\
& - 100a_6a_3a^2 - 900a_4a_3^2a - 30a_4a_6a_3 + 1080a_4a_3b + 270a_1a^3a_3 + 150a_4aa_6 \\
& + 44a_1ba_6 - 1350a_4a^2a_3 - 584a_1ba^2 + 28a_6ab + 198a_3a^2b - 22a_1a^2a_6 \\
& + 1440a_4ab + 24a_3^5 - 15a_3^3a^2 - 136ab^2 - 9a_3a^4 - 144a_3b^2 - 14a_6a^3 \\
& + 52a^3b + 272a_1b^2 - 540a_4a^3 + 30ba_3^3 - 52a_1a_3^4 + 26aa_3^4 + 97a_3a_6^2 - a^5 \\
& - 90a_4a_3^3 - 146a_6a_3^3.
\end{aligned}
$$

We have also computed the polynomials $w_7, \ldots, w_{13}$, but the expressions for these polynomials are too long. We do not present them here, however the interesting reader can easily compute them by (21) using any available computer algebra system.

Passing to the third step of the algorithm we eliminate from the system $v_2 = w_2, v_3 = w_3, \ldots, v_{13} = w_{13}$ the variables $c_1, \ldots, c_{11}$, that is, in the current notation, d, e, c, k, t, v, (note that any even equation $v_{2k} = w_{2k}$ is linear with respect to c_{2k-1}) and obtain

$$
\begin{aligned}
s_1 = {} & \left(a^2 - aa_1 + 10a_1^2 - 5aa_3 + 10a_1a_3 + 4a_3^2 - 9a_4 - 3a_6\right)/3 + b = 0, \\
s_2 = {} & -320a_1^4 - 1040/3a_1^3a_3 + 96a_1^2a_3^2 + 208a_1a_3^3 + 224/3a_3^4 - 112a_1^3a - 136a_1^2a_3a \\
& - 128a_1a_3^2a - 104a_3^3a - 8a_1^2a^2 + 16a_1a_3a^2 + 32a_3^2a^2 - 8/3a_3a^3 + 752a_1^2a_4 \\
& + 656a_1a_3a_4 + 32a_3^2a_4 + 88a_1^2a_6 + 96a_1a_3a_6 + 8a_3^2a_6 + 184a_1a_4a + 104a_3a_4a \\
& - 8a_1a_6a - 8a_3a_6a + 8a_4a^2 - 216a_4^2 - 48a_4a_6,
\end{aligned}
$$

and so on. Then, using the routine *minAssChar* of Singular [9] (which computes minimal associate primes of a polynomial ideal using the characteristic sets method [21]) we find that the variety of the ideal $\langle s_2, \ldots, s_6 \rangle$ consists of 11 components. To speed up computations we performed them in the ring of characteristic 32 003. Then, using the reconstruction to rational arithmetic we obtained nine conditions presented in the statement of the theorem, and the following two conditions:

$$a_6 = a_1 + \frac{5}{4}a_3 + \frac{1}{4}a = a_3 a - \frac{1}{7}a^2 - \frac{20}{7}a_4 = a^3 + \frac{35}{2}a_3 a_4 - \frac{9}{2}a_4 a$$
$$= a_3^2 + \frac{101}{59}a_3 a - \frac{6}{59}a^2 - \frac{67}{101}a_4 = 0 \tag{34}$$

and

$$a_4 = a_1 + \frac{8}{7}a_3 + \frac{1}{7}a = a_3 a - \frac{13}{8}a^2 + \frac{49}{2}a_6 = a^3 - \frac{32}{3}a_3 a_6 - \frac{44}{3}a_6 a$$
$$= a_3^2 + \frac{11}{69}a_3 a - \frac{169}{88}a^2 + \frac{53}{144}a_6 = 0. \tag{35}$$

Although the computations in modular arithmetic are very efficient they do not guarantee the correct result. In order to check the correctness of the obtained conditions we took the first two polynomial from each of the obtained series conditions, set them equal to zero and recompute with *minAssChar* of singular in the ring of characteristic zero (that is, for example, for the component defined by (35) we recomputed the minimal associate primes after the substitution $a_4 = 0, a_1 = -8/7a_3 - 1/7a$). The recalculations yield the same conditions as in the statement of the theorem, but instead of (34) and (35) we obtain

$$a_6 = 4a_1 + 5a_3 + a = a_3 a + a_3^2 - 4a_4 = a^2 - 2aa_3 + 5a_3^2$$
$$= 2a_3^3 + aa_4 - 3a_3 a_4 = 0 \tag{36}$$

and

$$a_4 = 7a_1 + 8a_3 + a = 11a_3 a - 52a_3^2 - 49a_6 = 48a^3 + 11aa_6 + 8a_6 a_3$$
$$= 11a^2 - 32a_3^2 - 196a_6 = 0. \tag{37}$$

It is easy to see that the only real solution for each of systems (36) and (37) is $a = a_1 = a_3 = a_4 = a_6 = 0$, so we should not take into account these conditions.

We now have to show that any system from components (1)–(9) defined in the statement of the theorem has the isochronous centre at the origin, in other words, that conditions (1)–(9) of theorem 3 are not only the necessary, but also the sufficient conditions for isochronicity of system (11). For case (2), the corresponding system of the form (3) is

$$\dot{x} = y, \qquad \dot{y} = -x(1 - ax) - \frac{5ay^2}{4(1 - ax)}. \tag{38}$$

If $a = 0$ then the system is linear, otherwise, after the transformation

$$x \mapsto x/a, \qquad y \mapsto y/a, \tag{39}$$

we obtain system (31) of the example of section 3. It is proven in section 3 that the centre at the origin of (31) is isochronous.

Simple calculations also show that (30) holds with $k_1 = 0$ for cases (1), (6) and (8), with $k_1 = a, k_2 = 2, k_3 = a^2 + 4a_6$ for case (7). In the remaining cases we rescale the phase plane coordinates using (39) and then for the obtained systems (30) is fulfilled with $k_1 = 3, k_2 = 4, k_3 = 9$ for case (3), with $k_1 = |k_2| \neq 0, k_3 = 0$ for case (4), with $k_1 = 2, k_2 = 3, k_3 = 4$ for case (5) and with $k_1 = 1, k_2 = 3, k_3 = 1$ for case (9). In all cases the corresponding functions (29) are the Urabe functions of these systems. □

Remark 1. To understand why in all isochronicity cases for system (11) the Urabe function can be chosen in the form (29) is an open problem.

Remark 2. As we have already mentioned in the introduction, the conditions of isochronicity given by theorem 3 are equivalent to those obtained in [3, 14]. Our approach based on theorem 2 is completely different from those of [3] and [14]. As we mentioned above, although we performed the calculations with singular in the field of the characteristic 32 003, after the rational reconstruction we obtained conditions (1)–(9) of theorem 3 which coincide with the conditions of [3] and [14].

The proof of the sufficiency of the conditions for isochronicity given above is very short, however it works only in the case when the Urabe function is of the form (29). We now present the way which we originally used in order to find the Urabe functions for all the cases of theorem 3. It appears that this approach can also be useful in the cases when the Urabe function is not of the form (29).

Consider again the second case of theorem 3, that is system (38). In the proof of theorem 3 we have computed the polynomials v_i and w_i. From (24) we now find that for system (31) the coefficients of the expansion (33) are

$$d = \frac{a}{4}, \qquad e = -\frac{a^3}{128}, \qquad c = \frac{3a^5}{8192}, \qquad k = -\frac{5a^7}{262\,144},$$
$$t = \frac{35a^9}{33\,554\,432}, \qquad v = -\frac{63a^{11}}{1073\,741\,824}.$$

Let $v_0 = d$, $v_1 = e$, $v_2 = c$ and so on. We see that for the terms of this sequence

$$\frac{v_{k+1}}{v_k} = -\frac{(2k+1)a^2}{32(k+1)} = -\frac{a^2}{16}\,\frac{(k+1/2)(k+1)}{(k+1)^2}. \tag{40}$$

Thus, we guess that (40) is the so-called hypergeometric sequence, and if so, then, using the algorithm from [12, p 36] (which, actually, follows from the definition of hypergeometric functions), we conclude

$${}_2F_1\left[\begin{matrix}1/2 & 1\\ 1 & \end{matrix}; \frac{-a^2x}{16}\right] = 1 + \sum_{k\geqslant 1} \frac{(2k-1)!!\cdot k!}{2^k\cdot k!}\cdot\frac{(-a^2x/16)^k}{k!} = \frac{4}{\sqrt{16+a^2x}}.$$

Since

$$\frac{4}{\sqrt{16+a^2x}} = 1 - \frac{a^2x}{32} + \frac{3a^4x^2}{2048} - \frac{5a^6x^3}{65\,536} + \cdots,$$

we obtain

$$\frac{ax}{\sqrt{16+a^2x^2}} = \frac{ax}{4}\,\frac{4}{\sqrt{16+a^2x^2}} = \frac{ax}{4} - \frac{a^3x^3}{128} + \frac{3a^5x^5}{8192} - \frac{5a^7x^7}{262\,144} + \cdots.$$

Thus, we guess for the second case of theorem 3 that the Urabe function is

$$h(\xi) = \frac{a\xi}{\sqrt{16+a^2\xi^2}}.$$

In the case $a = 1$ the latter function is the function (32).

Similarly, one can find the Urabe function in the form (29) for all other cases of theorem 3.

To finish our study of system (11) we mention the connection of families (1)–(9) of theorem 3 to some known isochronous potentials. For components (1), (6) and (8) of the theorem $h(\xi) \equiv 0$. Therefore, substitution (6) reduces systems from (1), (6) and (8) to the harmonic oscillator.

The function $h(\xi) = \xi$ leads to Urabe's potential

$$U(x) = 1 + x - \sqrt{1+2x}, \tag{41}$$

where $-1/2 < x < 3/2$, i.e. the potential is an analytic function defined on a finite segment of real axis. Substitution (6) brings systems from component (3) to systems of the form (7) with the potential (41).

The remaining cases arisen in our study of system (11) correspond to the function $h(\xi)$ of the form (29) with $k_1k_2k_3 \neq 0$. We can rewrite such functions as $h(\xi) = c_1\xi/\sqrt{1+c_2\xi^2}$. The rescaling $\xi\beta = \zeta$, where

$$\beta = c_2/c_1 = k_3/(k_1k_2), \tag{42}$$

yields $h(\zeta) = \alpha\zeta/\sqrt{1+\alpha\zeta^2}$. It is shown in [8] that the latter function gives rise to the potential

$$U(\alpha, x) = \frac{1}{2}\left(\frac{x+1-\sqrt{\alpha x(x+2)+1}}{1-\alpha}\right)^2 \tag{43}$$

studied in [2]. Using the scaling properties of $h(\xi)$ and the corresponding potentials [8, p 6187] we obtain from (43) the two-parameter family of isochronous potentials presented for the first time by Stillinger and Stillinger [8, 19]:

$$U(\alpha, \beta, x) = \frac{1}{2}\left(\frac{\beta x+1-\sqrt{\alpha\beta x(\beta x+2)+1}}{1-\alpha}\right)^2. \tag{44}$$

In the limit $\alpha \to 1$ (44) yields the isotonic potential

$$U(\beta, x) = \frac{1}{8\beta^2}\left(\beta x+1-\frac{1}{\beta x+1}\right)^2.$$

In terms of the constants k_1, k_2, k_3 of the function (29) we rewrite the latter potential as

$$U(k_1, k_2, k_3, x) = \frac{1}{8}\left(\frac{k_1k_2}{k_3}\right)^2\left(1+\frac{k_3x}{k_1k_2}-\frac{k_1k_2}{k_1k_2+k_3x}\right)^2.$$

Thus substitution (6) reduces systems from components (2), (3), (5), (7) and (9) to the isotonic potential. For example, for system (31) (which is a system from component (2) of theorem 3) the values of the parameters k_i are $k_1 = 1, k_2 = 4, k_3 = 1$; thus the corresponding potential appearing in the Hamiltonian (8) is

$$U(u) = \frac{u^2(8+u)^2}{8(4+u)^2}.$$

5. Final remarks

We have described in detail a new method (derived from Urabe's criterion) to compute the necessary conditions for isochronicity of periodic solutions within finite-parametric families of system (1) (provided such families can be transformed to the Liénard-type system (3). We have applied it to study the isochronicity of system (11) and have found all isochronous systems in this family. We have obtained the explicit expressions for the Urabe functions for all these isochronous systems, which in all cases are functions of the form (29). An interesting open question naturally arises in this connection whether there are polynomial systems reducible to (3) with the Urabe function different from (29).

In fact the study of the isochronicity problem for parametric families of ODEs consists of two parts: one is a difficult computational problem (as we have seen above) to find the necessary conditions for isochronicity, and then the problem arises to prove that the obtained conditions are also the sufficient conditions for isochronicity. No general methods to treat this second problem are known. The three main (but not universal) methods are the construction of Darboux linearization, the construction of an orthogonal commuting system and the direct computation of the period function in the polar coordinates. In this paper, we have demonstrated for the first time that the Urabe criterion also is a very efficient method to check isochronicity of some parametric families of systems of ODEs.

Note also, that despite the fact that the proof of theorem 3 looks rather simple, in fact it requires tremendous computational work: we have to find the decomposition of the solution of the polynomial system $s_2 = s_3 = \cdots = s_6 = 0$, where the polynomials $s_2, \ldots, s_6$ have 26, 79, 174, 335 and 587 terms, respectively. We are able to carry out this work just because we have used one of the most modern and efficient softwares available for such purposes (the routines of Singular [9]) and the trick with the modular arithmetics.

As a byproduct of our study and as a complement to the classification of polynomial systems presented in [5] in the appendix we give linearizing substitutions for two isochronous families of (11).

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Appendix

In recent years the isochronicity problem for the cubic system

$$\dot{x} = -y + \sum_{i+j=2}^{3} a_{ij} x^i y^j, \qquad \dot{y} = x + \sum_{i+j=2}^{3} b_{ij} x^i y^j \tag{A.1}$$

has been tackled by many authors (see e.g. [4, 5, 16] and references therein). In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ we can write reversible systems (A.1) as

$$\begin{aligned} \dot{r} &= r^2(R_3 \sin 3\theta + R_1 \sin\theta) + r^3(R_4 \sin 4\theta + R_2 \sin 2\theta), \\ \dot{\theta} &= 1 + r(R_3 \cos 3\theta + r_1 \cos\theta) + r^2(R_4 \cos 4\theta + R_2 \cos 2\theta + r_0). \end{aligned} \tag{A.2}$$

The classification presented in [4] is for the case $R_3 = 0$. Note that for system (11)

$$R_3 = (a_1 - a_3 + a)/4, \tag{A.3}$$

thus this class is different from the one studied in [4].

We have observed that some systems from theorem 3 are systems from the classification given in [5]. Namely, they are the systems from components (5), (7) and (8) of theorem 3 (for all these cases $R_3 = 0$).

For case (5), the corresponding differential system is linear or after the rescaling of x and y can be written as

$$\dot{x} = -y + xy - \tfrac{2}{9}x^2 y, \qquad \dot{y} = x - \tfrac{2}{3}x^2 + \tfrac{1}{3}y^2 + \tfrac{2}{9}x^3. \tag{A.4}$$

This is the system CR_5 given on page 41 in [5].

System (A.4) can be linearized, that is, can be brought to the linear system $\dot{X} = -Y$, $\dot{Y} = X$ by the transformation

$$X = \frac{(3-x)(9x - 6x^2 + 2x^3 - 6y^2 + 2xy^2)}{3(2x-3)^2}, \qquad Y = \frac{3y(3-x)}{(2x-3)^2}. \tag{A.5}$$

Systems from the component (7) are of the form

$$\dot{x} = -y + axy + a_6x^2y, \qquad \dot{y} = x - \tfrac{1}{2}ax^2 + \tfrac{1}{2}ay^2 + a_6xy^2. \tag{A.6}$$

This is the system CR_1 of [5, p 38]. After the substitution

$$u = x + \mathrm{i}y, \qquad v = x - \mathrm{i}y, \tag{A.7}$$

system (A.6) is written in the complex form as

$$\dot{u} = u - \frac{a}{2}u^2 - \frac{a_6}{4}u^3 + \frac{a_6}{4}uv^2, \qquad \dot{v} = -v + \frac{a}{2}v^2 - \frac{a_6}{4}u^2v + \frac{a_6}{4}v^3. \tag{A.8}$$

The latter system is case VIII in [16], where the linearizing transformation is given. For $a_6 \neq -a^2/4$ the linearization is

$$u_1 = u\ell_1^{\alpha_1}\ell_2^{\alpha_2}\ell_3^{\alpha_3}, \qquad v_1 = v\ell_1^{\alpha'_1}\ell_2^{\alpha'_2}\ell_3^{\alpha'_3},$$

where

$$\ell_1 = 1 - \frac{a + \sqrt{a^2 + 4a_6}}{4}u - \frac{a - \sqrt{a^2 + 4a_6}}{4}v,$$
$$\ell_2 = 1 - \frac{a + \sqrt{a^2 + 4a_6}}{4}u - \frac{a + \sqrt{a^2 + 4a_6}}{4}v,$$
$$\ell_3 = 1 - \frac{a - \sqrt{a^2 + 4a_6}}{4}u - \frac{a - \sqrt{a^2 + 4a_6}}{4}v,$$

and

$$\alpha_1 = -\frac{a}{\sqrt{a^2 + 4a_6}}, \qquad \alpha_2 = \frac{a - \sqrt{a^2 + 4a_6}}{2\sqrt{a^2 + 4a_6}}, \qquad \alpha_3 = \alpha_2,$$
$$\alpha'_1 = \frac{a}{\sqrt{a^2 + 4a_6}}, \qquad \alpha'_2 = -\frac{a + \sqrt{a^2 + 4a_6}}{2\sqrt{a^2 + 4a_6}}, \qquad \alpha'_3 = \alpha'_2.$$

Reverting to the real variables x, y we obtain the linearization

$$X = \operatorname{Re} u_1(x + \mathrm{i}y, x - \mathrm{i}y), \qquad Y = \operatorname{Im} u_1(x + \mathrm{i}y, x - \mathrm{i}y). \tag{A.9}$$

In the particular case when α_1 is real we can write this substitution as

$$X = \frac{1}{2}(1 - ax - a_6x^2)^{(-1-\alpha_1)/2}\left((x + \mathrm{i}y)\left(1 - \frac{ax}{2} + \frac{\mathrm{i}ay}{2\alpha_1}\right)^{\alpha_1} + (x - \mathrm{i}y)\left(1 - \frac{ax}{2} - \frac{\mathrm{i}ay}{2\alpha_1}\right)^{\alpha_1}\right),$$
$$Y = -\frac{\mathrm{i}}{2}(1 - ax - a_6x^2)^{(-1-\alpha_1)/2}\left((x + \mathrm{i}y)\left(1 - \frac{ax}{2} + \frac{\mathrm{i}ay}{2\alpha_1}\right)^{\alpha_1} - (x - \mathrm{i}y)\left(1 - \frac{ax}{2} - \frac{\mathrm{i}ay}{2\alpha_1}\right)^{\alpha_1}\right).$$

If $a_6 = -a^2/4$, then system (A.8) is linearized by the transformation

$$u_1 = \frac{4\,\mathrm{e}^{\frac{a(-u+v)}{-4+a(u+v)}}\,u}{4 - a(u+v)}, \qquad v_1 = \frac{4\,\mathrm{e}^{\frac{a(u-v)}{-4+a(u+v)}}\,v}{4 - a(u+v)},$$

and the change

$$X = \frac{2\left(x\cos\left(\frac{ay}{ax-2}\right) + y\sin\left(\frac{ay}{ax-2}\right)\right)}{2-ax},$$
$$Y = \frac{2\left(y\cos\left(\frac{ay}{ax-2}\right) - x\sin\left(\frac{ay}{ax-2}\right)\right)}{2-ax} \tag{A.10}$$

yields the linear system in the case of real system (A.6).

Note also that system of case (8) is the cubic reversible system CR_2 of [5, p 38]. A linearization is provided there.

For most of the systems of the classification in [5] the authors provide the linearizing transformations; however for systems (A.4) and (A.6) such transformations are not presented. Thus, linearizations (A.5), (A.9) and (A.10) are a complement to the classification of cubic systems given in [5].

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